2011

TRIAL

HIGHER SCHOOL CERTIFICATE

GIRRAWEEN HIGH SCHOOL

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board – approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

Attempt Questions 1 – 8

All questions are of equal value

Total marks-120

Attempt all 8 questions

All questions are of equal value

Answer each question on a separate piece of paper clearly marked Question 1, Question 2, etc.

Each piece of paper should show your name.

Question 1 (15 Marks) Use a separate piece of paper **Marks**

(a) Find $\int \frac{x^2}{x^2+1}\,dx$ **2**

(b) Find $\int \frac{x}{\sqrt{1-x}}\,dx$ **2**

(c) Evaluate $\int_0^{\frac{\pi}{4}} \sec^4 x \tan^3 x\,dx$ **3**

(d) Show by integration that $\int \frac{1-\sin x}{\cos^2 x}\,dx = \frac{\sin x - 1}{\cos x} + c$ **2**

(e) Using the substitution $t = \tan\frac{x}{2}$ or otherwise, evaluate $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{dx}{1-\cos x}$ **3**

(f) Use integration by parts to find $\int \sin^{-1} x\,dx$ **3**

Marks

Question 2 (15 Marks) Use a separate piece of paper

(a) Given that $z = 2 - i$ and $w = 3 - 4i$ express in the form $x+iy$

(i) z^2 **1**

(ii) $\frac{2}{z}$ **1**

(iii) $\frac{w}{i}$ **1**

(b) Given that $z = (\sqrt{3} - 1) + (\sqrt{3} + 1)i$

(i) Express z in modulus –argument form **2**

(ii) Express z^4 in modulus –argument form **1**

(iii) Express z^4 in the form $a + bi$ **1**

(c) Given that $z^5 = i$

(i) Solve for z and sketch your solution on an Argand diagram. **3**

(d) Sketch the region in the Argand plane where $|z - 1 - 2i| \le 2$

and $0 \le \arg(z-1) \le \frac{\pi}{2}$ hold simultaneously **3**

(e) Given that $|z| = 1$ and that $0 \le \arg z \le \frac{\pi}{2}$. Show that $2\arg(z+1) = \arg z$ **2**

Marks

Question 3 (15 Marks) Use a separate piece of paper

(a) The ellipse $\mathscr{E}$ given by the equation $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{9} = 1$

(i) Find the eccentricity of $\mathscr{E}$ **2**

(ii) Find the coordinates of the foci S, S' and the equation of the directrices **2**

(iii) Find the points of intersection with the coordinate axes. **2**

(iv) Sketch the ellipse showing all important features **3**

(b) The hyperbola $\mathscr{H}$ is given by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ it has foci S and S'

(i) Show that the equation of the tangent T to $\mathscr{H}$ at P $(a\sec\theta, b\tan\theta)$

is $\frac{\sec\theta}{a}x - \frac{\tan\theta}{b}y = 1$ **3**

(ii) Find G the point of intersection between the tangent and the x-axis. **1**

(iii) Show that $\frac{PS}{PS'} = \frac{GS}{GS'}$ **2**

Marks

Question 4 (15 Marks) Use a separate piece of paper

(a) A 100gm bullet is fired vertically into the air from the ground with an initial velocity of 1200m/s. The bullet is subject to air resistance of $\frac{mv}{g}$ newtons in the opposite direction to the motion. The bullet is also subject to a downwards gravitational force of mg newtons. Assume $g = 10m/s^2$

(i) Find an expression for the equation of motion. ($\ddot{x}$ in terms of v) 1

(ii) Find the time taken to reach maximum height 2

(iii) Find the maximum height. 2

(iv) After reaching maximum height the bullet returns to the ground. Find the terminal velocity. 1

(b) A train of mass 20 tonnes is rounding a curve of radius 200m at a speed of 72k/h.

(i) Find the horizontal thrust on the outer wheels of the train. Assume $g = 10ms^{-2}$ 2

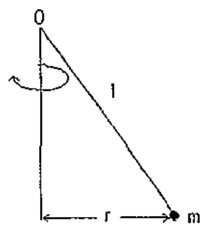
(ii) At what angle should the track be angled to eliminate this thrust, 2

(iii) Representing the train by a point mass M draw a diagram to represent all forces in part (ii) if the train is travelling at less than 72k/h. 1

(iv) The train now rounds the angled track at 36k/h find the thrust on the inner wheels 2

(c) A particle of mass m is connected to a fixed point O by a light inextensible string of length l. The particle describes a horizontal circle of radius r whose centre is vertically below O with velocity v. If the semi vertical angle is α, resolve forces vertically and horizontally to show that

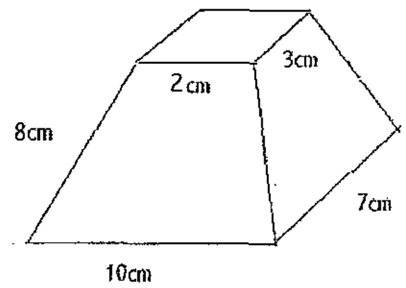
$$v = \sqrt{lg\sin\alpha\tan\alpha}$$ 2



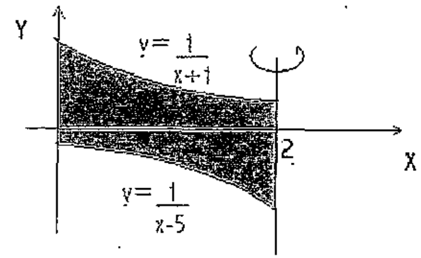
Marks

Question 5 (15 Marks) Use a separate piece of paper

(a) The diagram below represents a solid block of height 8cm, the base is 10cm by 7cm the top has dimensions 2cm by 3cm. The planes containing the base and the top are parallel. By taking slices parallel to the base determine the volume of the pyramid. 3



(b) The region between the curves $y = \frac{1}{x+1}$ and $y = \frac{1}{x-5}$ and the Y axis and the line x = 2 is rotated about the line x = 2.



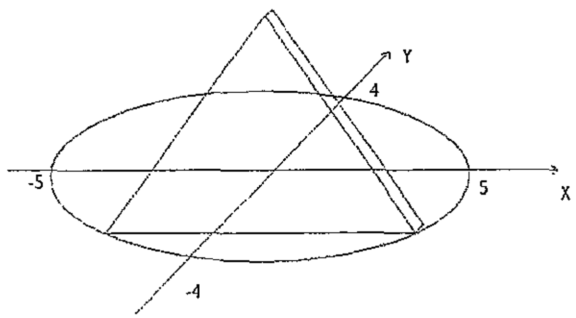
(i)Using the method of cylindrical shells find the volume δV of a representative shell 2

(ii)Find the volume of the resulting solid. 3

Marks

Question 5 continued.

(c) The base of a certain solid is the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$. Each plane section of this solid cut out by a plane perpendicular to the y-axis is an equilateral triangle with one side in the ellipse. Find the volume of this solid. **3**



(d) The point $P(p,\frac{1}{p})$ lies on the rectangular hyperbola $xy=1$.

(i) Find the equation of the normal from P. **2**

(ii) The normal intersects the hyperbola at Q. Find the coordinates of Q in terms of p. **2**

Marks

Question 6 (15 Marks) Use a separate piece of paper

(a) The cubic equation $2x^3-9x^2+14x-5=0$ contains the root $2+i$
Stating all necessary assumptions find all factors with real coefficients of the cubic. **3**

(b) (i) The polynomial $P(x)$ has a root of multiplicity m prove that $P'(x)$ has the same root, of multiplicity $m-1$. **2**

(ii) The polynomial $P(x)=ax^3+bx^2+cx+d=0$ has a double root α.
Show that if α is real $b^2\geq 3ac$ **2**

(iii) Show also that $\alpha = \frac{-c\pm\sqrt{c^2-3bd}}{b}$ **2**

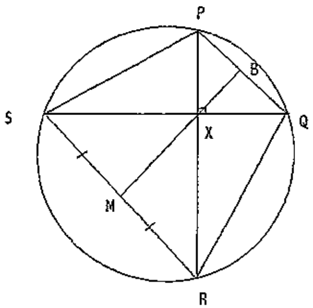
(c) The ellipse $9y^2+x^2-3xy=6$ is known to have only two stationary points.
Verify that the these point occur when $x=\pm\sqrt{2}$ **2**

(d) The fibonacci sequence 1, 1, 2, 3, 5, 8, ...
May be defined as $t_1=t_2=1$ and $t_n=t_{n-1}+t_{n-2}$
Prove by Mathematical Induction that

$t_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}$ where $\alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}$ **4**

Marks

Question 7. (15 Marks) Use a separate piece of paper



(a) In the circle above SPQR is a cyclic quadrilateral. The diagonals PR and SQ are perpendicular meeting at X. The line from the midpoint M of SR through X meets PQ at B.
Prove that MB is perpendicular to QP. 4

(b) $I_n = \int_0^1 x^n \sqrt{1-x}\,dx$

(i) Show that $I_n = \dfrac{2n}{2n+3} I_{n-1}$ 2

(ii) Hence evaluate $\int_0^1 x^3 \sqrt{1-x}\ dx$, 2

(iii) Show that $I_n = \dfrac{n!(n+1)!}{(2n+3)!} 4^{n+1}$ 3

(c) In a raffle there are 100 tickets each with three digits, numbered from 001 to 100.
Four tickets are drawn for four prizes.

(i) What is the probability that the first ticket drawn contains a repeated digit. 1

(ii) What is the probability that the first ticket does not contain the digit 7. 1

(iii) Find the probability that the four winning tickets are drawn with the tickets in ascending order. 2

Marks

Question 8. (15 Marks) Use a separate piece of paper

(a) Let $x = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$ and $b = x + \dfrac{1}{x}$, $b \geq 0$

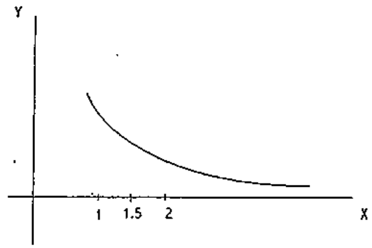
(i) Show that $x^4 + x^3 + x^2 + x + 1 = 0$ 1

(ii) Use part (i) to show that $b^2 + b - 1 = 0$ 1

(iii) Show that $x^2 - bx + 1 = 0$ and hence that $x = \dfrac{b + i\sqrt{4-b^2}}{2}$ 2

(iv) Hence show that $\cos\dfrac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$ 2

(b) (i) Evaluate $\int_1^2 \dfrac{1}{x} dx$ 1



(ii) Considering the upper and lower rectangles of width 0.5 show that

$\dfrac{7}{12} \leq \int_1^2 \dfrac{1}{x}\,dx \leq \dfrac{5}{6}$ 1

(iii) Explain why $\dfrac{1}{n+1} \leq \int_n^{n+1} \dfrac{1}{x}\,dx \leq \dfrac{1}{n}$ 1

(iv) Hence deduce that $(1+\dfrac{1}{n})^n \leq e \leq (1+\dfrac{1}{n})^{n+1}$ 2

(c) In how many ways can the letters of the word MMAATTHH be arranged

(i) Without restriction 1

(ii) With at least one pair of repeated letters together 1

(iii) With no pair of repeated letters together. 2

END OF PAPER

Question 1

EXT 2 TRIAL HSC 2011

(a) $\int \frac{x^2}{x^2+1}\,dx$

$= \int \frac{x^2+1-1}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}\,dx$

$= x - \tan^{-1}x + C$ (2)

(b) $I = \int \frac{x}{\sqrt{1-x}}\,dx$

let $u = 1-x$ $\quad x = 1-u$

$\frac{du}{dx} = -1$

$du = -dx$

$I = \int \frac{(1-u)\cdot du}{\sqrt{u}}$

$= \int u^{1/2} - u^{-1/2}\,du$

$= \frac{2}{3}u^{3/2} - 2u^{1/2} + C$ (2)

$= \frac{2}{3}(1-x)^{3/2} - 2\sqrt{1-x} + C.$

(c) $I = \int_0^{\pi/4} \sec^4 x \tan^3 x\,dx$

let $u = \tan x$ $\quad du = \sec^2 x\,dx$

$x=0$ $u=0$ $\quad x=\pi/4$ $u=1$

$I = \int_0^{\pi/4} \sec^2 x \tan^3 x \sec^2 x\,dx$

$= \int_0^1 (u^2+1)\,u^3\,du$

$= \int_0^1 u^5 + u^3\,du$

$= \left[\frac{u^6}{6} + \frac{u^4}{4}\right]_0^1$

$= \frac{1}{6} + \frac{1}{4}$

$I = \frac{5}{12}$ (3)

(d) $\int \frac{1-\sin x}{\cos^2 x}\,dx$

$= \int \sec^2 x\,dx - \int \frac{\sin x\,dx}{\cos^2 x}$

$= \tan x + \int \frac{-\sin x\,dx}{\cos^2 x}$

let $u = \cos x$

$du = -\sin x\,dx$

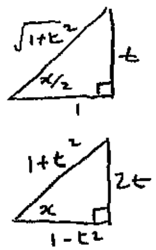
$= \tan x + \int \frac{du}{u^2} + c$

$= \tan x - \frac{1}{u} + c$

$= \tan x - \frac{1}{\cos x}$

$= \frac{\sin x - 1}{\cos x} + C$ (2)

(e) $t = \tan\frac{x}{2}$



$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(\tan^2\frac{x}{2} + 1\right)$

$\frac{dt}{dx} = \frac{1}{2}(t^2+1)$

$\frac{dx}{dt} = \frac{2}{t^2+1}$

$dx = \frac{2dt}{t^2+1}$

$\cos x = \frac{1-t^2}{1+t^2}$

when $x = \pi/2$ $\quad t = 1$

$x = 2\pi/3$ $\quad t = \sqrt{3}$

$\therefore I = \int_{\pi/2}^{2\pi/3} \frac{dx}{1-\cos x}$

$= \int_1^{\sqrt{3}} \frac{\frac{2dt}{t^2+1}}{1-\frac{(1-t^2)}{1+t^2}}$

$= \int_1^{\sqrt{3}} \frac{\frac{2dt}{1+t^2}}{\frac{1+t^2-(1-t^2)}{1+t^2}}$

$= \int_1^{\sqrt{3}} \frac{2dt}{2t^2}$

$= \int_1^{\sqrt{3}} \frac{dt}{t^2}$

$= \left[-\frac{1}{t}\right]_1^{\sqrt{3}}$

$= \frac{-1}{\sqrt{3}} + 1$

$= \frac{\sqrt{3}-1}{\sqrt{3}}.$ (3)

(f) $I = \int 1\cdot\sin^{-1}x\,dx$

let $v = \sin^{-1}x\,dx$ $\quad du = 1$

$\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\quad u = x$

$\int v\frac{du}{dx} = uv - \int u\cdot\frac{dv}{dx}$

$\int \sin^{-1}x\cdot(1) = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$ (3)

$= x\sin^{-1}x + \sqrt{1-x^2} + C$

Question 2.

(a) $z = 2 - i \quad \omega = 3 - 4i$

(i) $z^2 = (2-i)(2-i)$
$= 4 + 1 - 4i$
$= 3 - 4i = \omega.$ (1)

(ii) $\frac{2}{z} = \frac{2}{2-i} \cdot \frac{2+i}{2+i} = \frac{4+2i}{5}$
$= \frac{4}{5} + \frac{2}{5}i$ (1)

(iii) $\frac{\omega}{i} = \frac{3-4i}{i} \times \frac{i}{i} = \frac{3i+4}{-1}$
$= -4 - 3i$ (1)

(b) $z = \sqrt{3} - 1 + (\sqrt{3}+1)i$

(i) $\text{mod } z = \tan^{-1} \frac{\sqrt{3}+1}{\sqrt{3}-1}$
$= \frac{5\pi}{12}$
$\arg z = \sqrt{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}$
$= \sqrt{8} = 2\sqrt{2}$

$z = 2\sqrt{2} \text{ cis } \frac{5\pi}{12}$ (2)

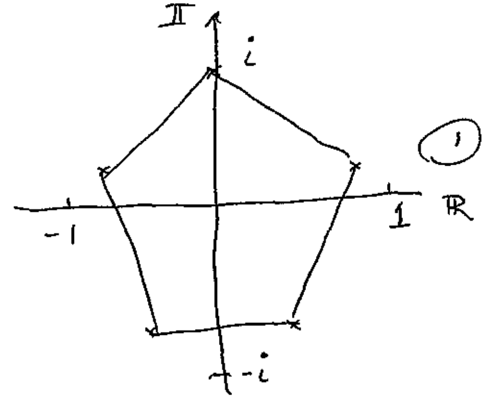
(ii) $z^4 = (2\sqrt{2})^4 \text{ cis } 4\left(\frac{5\pi}{12}\right)$
$= 64 \text{ cis } \frac{5\pi}{3}$ (1)
$= 64 \text{ cis } -\frac{\pi}{3}$

(iii) $z^4 = 64\left(\cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3}\right)$
$= 64\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$= 32 - 32\sqrt{3}\,i$ (1)

(c) $z^5 = i$

(i) $z^5 = \text{cis } \frac{\pi}{2} + 2k\pi \quad k = -2, -1, 0, 1, 2$

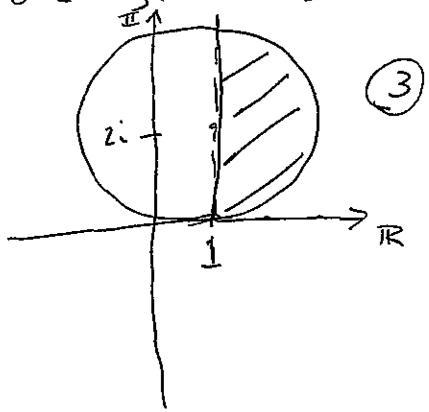
$z = \text{cis} -\frac{7\pi}{10}, \text{ cis} -\frac{3\pi}{10}, \text{ cis } \frac{\pi}{10}$



(1)

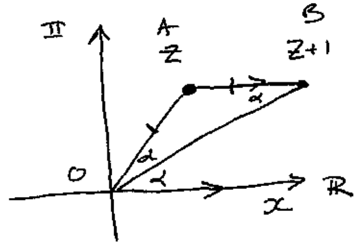
(d) $|z - (1+2i)| \le 2$

$0 \le \arg(z-1) \le \frac{\pi}{2}$



(3)

(e)



IN $\triangle OAB$
$OA = AB$
$\therefore \angle AOB = \angle ABO$
But $AB \parallel OX$
$\therefore \angle BOX = \angle ABO$
$\therefore \angle AOX = 2\angle BOX$
$\therefore \arg z = 2 \arg(z+1).$ (2)

Question 3.

(a) (i) $\frac{(x-3)^2}{5^2} + \frac{(y-2)^2}{3^2} = 1$

$b^2 = a^2(1-e^2)$
$9 = 25(1-e^2)$
$\frac{9}{25} = 1 - e^2$ 1
$e^2 = \frac{16}{25}$ 1
$e = \frac{4}{5}$ (2)

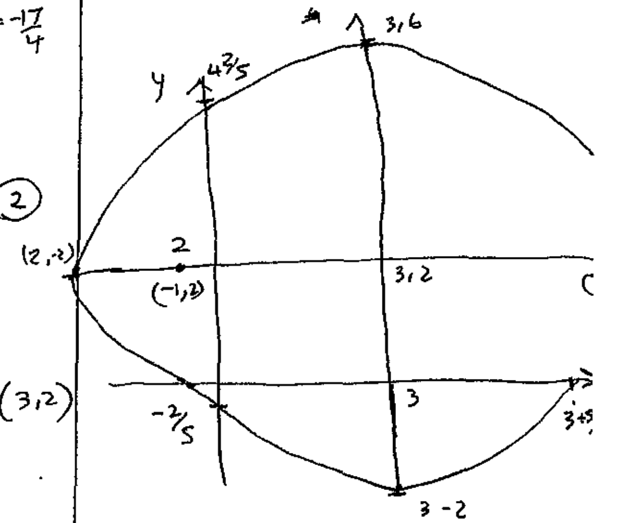
$y = -\frac{17}{4}$

(ii) $S(\pm ae, 0)$ 1
$S(\pm 4, 0)$ 1
Centre of parabola $(3,2)$
$S(7,2) \quad S'(-1,2)$
Directrix $X = \pm \frac{a}{e}$
$X = \pm \frac{25}{4}$
Centre $(3,2)$
$X = -\frac{13}{4} \quad X = \frac{37}{4}$ (2)

(iii) X AXIS $Y = 0$
$\frac{(x-3)^2}{25} + \frac{4}{9} = 1$
$\frac{(x-3)^2}{25} = \frac{5}{9}$
$(x-3)^2 = \frac{125}{9}$
$x - 3 = \pm \frac{5}{3}\sqrt{5}$
$x = 3 \pm \frac{5}{3}\sqrt{5}.$

Y AXIS $x = 0$
$\frac{9}{} + (x\ \ )= 1$

$\frac{(x-2)^2}{9} = \frac{16}{25}$
$(x-2)^2 = \frac{144}{25}$
$y - 2 = \pm \frac{12}{5}$
$y = 2 \pm \frac{12}{5}.$ (2)



Q5 cont.

b)

(i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

IMPLICIT DIFFERENTIATION

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = \frac{2x}{a^2}$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}.$$

EQ^N OF TANGENT

$$y - b\tan\theta = \frac{b^2x_1}{a^2y_1}(x - a\sec\theta)$$

$$y - b\tan\theta = \frac{b^2(a\sec\theta)}{a^2(b\tan\theta)}(x - a\sec\theta)$$

$$\frac{\tan\theta\, y - b^2\tan^2\theta}{b^2} = \frac{a\sec\theta\, x - a^2\sec^2\theta}{a^2}$$

$$\sec^2\theta - \tan^2\theta = \frac{a\sec\theta\, x}{a^2} - \frac{b\tan\theta\, y}{b^2}$$

$$1 = \frac{\sec\theta\, x}{a} - \frac{\tan\theta\, y}{b^2} \quad (3)$$

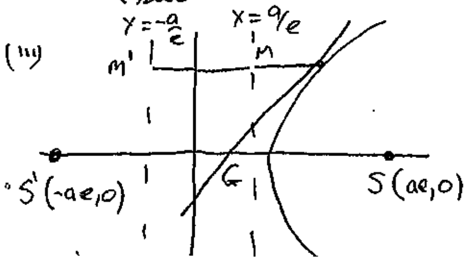
(ii) At x axis $y = 0$.

$$\frac{\sec\theta x}{a} = 1$$

$$x = \frac{a}{\sec\theta}$$

$$G\left(\frac{a}{\sec\theta}, 0\right).$$

(iii)



NOTING THAT

$PS = ePM \qquad PS' = ePM'$

$PM = a\sec\theta - \dfrac{a}{e} \qquad PM' = a\sec\theta + \dfrac{a}{e}$

$$\frac{PS}{PS'} = \frac{a(e\sec\theta - 1)}{a(e\sec\theta + 1)}$$

$GS = ae - \dfrac{a}{\sec\theta} \qquad GS' = ae + \dfrac{a}{\sec\theta}$

$$\frac{GS}{GS'} = \frac{a\left(\frac{e\sec\theta - 1}{\sec\theta}\right)}{a\left(\frac{e\sec\theta + 1}{\sec\theta}\right)}$$

$$= \frac{PS}{PS'} \quad (2)$$

Question 4

i) $F = ma$

$$ma = -\frac{mv}{g} - mg$$

$$\ddot{x} = -\left(\frac{v}{g} + g\right) \quad g = 10$$

$$= -\left(\frac{v}{10} + 10\right). \quad (1)$$

iii) at MAX HEIGHT $V = 0$

$$v\frac{dv}{dx} = -\frac{v}{10} - 10$$

$$v\frac{dv}{dx} = -\left(\frac{v+100}{10}\right)$$

$$\frac{dv}{dx} = -\left(\frac{v+100}{10v}\right)$$

$$\frac{dx}{dv} = \frac{-10v}{v+100}.$$

$$\frac{dx}{dv} = -10\left(\frac{v+100-100}{v+100}\right)$$

$$= -10\left(1 - \frac{100}{v+100}\right)$$

$$\int dx = -10\int 1 - \frac{100}{v+100}\,dv$$

$$X = +10\Big[V - 100\ln(V+100)\Big]_0^{1200}$$

$$X = 10\Big[(1200 - 100\ln 1300) - (0 - 100\ln 100)\Big]$$

$$X = 12000 - 1000\ln 13$$

$$X = 9.1125 \text{ km} \quad (2)$$

#ii) $x = -\dfrac{v+100}{10}$

$$\frac{dv}{dt} = -\frac{v+100}{10}$$

$$\frac{dt}{dv} = \frac{-10}{v+100}$$

$$t = -10\int_{1200}^{0}\frac{dv}{v+100}$$

$$t = 10\int_0^{1200}\frac{dv}{v+100}$$

$$t = 10\Big[\ln|v+100|\Big]_0^{1200}$$

$$t = 10\Big[\ln 1300 - \ln 100\Big]$$

$$t = 25.65 \text{ sec.} \quad (2)$$

iv) $F = ma$

$$ma = mg - \frac{mv}{g}$$

$$a = \frac{g^2 - v}{g}$$

FOR TERMINAL VEL $a = 0$

$$g^2 = v$$

$$v = 100 \text{ m/s}. \quad (1)$$

Q4

(b) (i) $F = \frac{mv^2}{r}$

72 km/hr = 20 m/s

20 T = 20,000 kg

$F = \frac{20{,}000 \times 20^2}{200}$

$F = 40{,}000$ N (2)

(ii) IF $F = 0$

$N \sin\theta = \frac{mv^2}{r}$

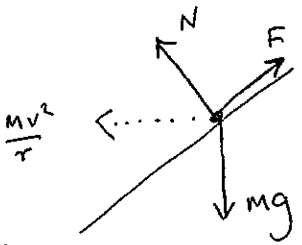
$N \cos\theta = mg$

$\tan\theta = \frac{v^2}{gr}$

$= \frac{20^2}{10 \times 200}$

$\theta = 11° 18.6'$ (2)

(iii)



(1)

(iv) RESOLVING HORIZONTALLY

$N \sin\theta - F\cos\theta = \frac{mv^2}{r}$ (1)

Resolving Vertically (2)

$N = \frac{20\,000 \times 10}{\cos 11° 18.6'}$

$N = 203{,}960.8$ Newton

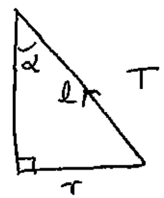
SUBST (1)

$F\cos\theta = N\sin\theta - \frac{mv^2}{r}$

$F\cos\theta = 30000$ N

$F = 30594.4$ N. (2)

(c)



$\sin\alpha = \frac{r}{l}$

$r = l\sin\alpha$

Resolving vertically

$T\cos\alpha = mg$ (1)

Resolving horizontally

$T\sin\alpha = \frac{mv^2}{r}$ (2)

(2) ÷ (1) $\tan\alpha = \frac{v^2}{rg}$

$v^2 = rg\tan\alpha$

$v^2 = gl\sin\alpha\tan\alpha$

$v = \sqrt{gl\sin\alpha\tan\alpha}$ (2)

Question 5.

(a) A Strip at height h

has length $10 - h$

has width $7 - \frac{h}{2}$

$\delta V = (10-h)(7-\frac{h}{2})\,\delta h$

$V = \lim_{\delta h \to 0} \sum_{h=0}^{8} 70 - 12h + \frac{h^2}{2}\,\delta h$

$V = \int_0^8 \frac{h^2}{2} - 12h + 70\, dh$

$= \left[\frac{h^3}{6} - 6h^2 + 70h\right]_0^8$

$= \frac{784}{3}\ u^3.$ (3)

(b) (i) V of cylindrical shell

$\delta V = \pi R^2 h - \pi r^2 h$

$= \pi\left\{(2-x)^2 - [2-(x+\delta x)]^2\right\}\left(\frac{1}{x+1} - \frac{1}{x-5}\right)$

$= \pi\left\{4 - 4x + x^2 - \left[4 - 4(x+\delta x) + (x+\delta x)^2\right]\right\}$

$\times \left(\frac{x-5-(x+1)}{(x+1)(x-5)}\right)$

$= \pi\left\{4\delta x - 2x\delta x - \delta x^2\right\}\frac{-6}{x^2-4x-5}$

$= 6\pi \frac{(4 - 2x + \delta x)\,\delta x}{5 + 4x - x^2}$ (2)

(ii) $Vol = 6\pi \lim_{\delta x \to 0} \sum_{x=0} \frac{4-2x}{5+4x-x^2}$

$V = 6\pi \int_0^2 \frac{4-2x}{5+4x-x^2}\, dx$

$V = 6\pi \left[\ln|5+4x-x^2|\right]_0^2$

$V = 6\pi (\ln 9 - \ln 5)$

$V = 6\pi \ln 1.8$ (3)

(c) at any value y:

Area $\Delta = \frac{1}{2} bh$

$= \frac{1}{2}(2x)\sqrt{3}x$

$\delta Vol = \sqrt{3}\, x^2\, \delta y$

$Vol = \lim_{\delta y \to 0} 2\sum_{y=0}^{4} \sqrt{3}\, x^2\, \delta y$

$V = 2\sqrt{3}\int_0^4 x^2\, dy$

$V = 2\sqrt{3}\frac{(25)}{16}\int_0^4 16 - y^2\, dy$

$\frac{x^2}{25} = \frac{16 - y^2}{16}$

$x^2 = \frac{25}{16}(16 - y^2)$

$$V = \frac{25\sqrt{3}}{8}\int_0^4 16 - y^2\,dy$$

$$V = \frac{25\sqrt{3}}{8}\left[16y - \frac{y^3}{3}\right]_0^4$$

$$V = \frac{25\sqrt{3}}{8}\left[\left(64 - \frac{64}{3}\right) - (0)\right]$$

$$= \frac{25 \times 128 \times \sqrt{3}}{24}\ u^3$$

$$V \doteqdot \frac{400\sqrt{3}}{3}\ u^3 \quad 3$$

b) (i) $y = 1/x$

$$\frac{dy}{dx} = -1/x^2$$

$$\frac{dy}{dx}(x=p) = -1/p^2$$

$\therefore$ GRADIENT OF NORMAL

$$= p^2$$

EQ^N OF NORMAL

$$y - \frac{1}{p} = p^2(x - p)$$

$$y = p^2x - p^3 + \frac{1}{p} \quad 2$$

OR $p^3x - py - p^4 + 1 = 0$

IF $y = 1/x$

$$p^3x - \frac{p}{x} - p^4 + 1 = 0$$

$$p^3x^2 - p - p^4x + x = 0$$

$$p^3x^2 - p^4x + x - p = 0$$

$$p^3x(x-p) + 1(x-p) = 0$$

$$(p^3x+1)(x-p) = 0$$

$x = p$ original

$x = -1/p^3 \quad 2$

$$\boxed{Q\left(q, \frac{1}{q}\right) = \left(\frac{-1}{p^3}, -p^3\right)}$$

Question 6.

(a) $2x^3 - 9x^2 + 14x - 5 = 0$

has a root $2+i$

$\therefore$ as coefficients are real

$2-i$ is also a root

Now $2+i+2-i+\alpha = -b/a$

$$4 + \alpha = 9/2$$

$$\alpha = 1/2$$

$(2x-1)$ IS A FACTOR

Leading coefficient is $\frac{2}{2}$

check $(2+i)(2-i)\frac{1}{2} = -d/a$

$$(5)\tfrac{1}{2} = 5/2 \quad \checkmark\checkmark$$

$$2x^3 - 9x^2 + 14x - 5 \quad 3$$

$$= (2x-1)(x^2 - 4x + 5)$$

$$(x - (2+i))(x - (2-i))$$

$$= x^2 - 4x + 5.$$

(b) (i) $P(x) = Q(x)(x-\alpha)^m$

$$P'(x) = Q'(x)(x-\alpha)^m + Q(x)\,m(x-\alpha)^{m-1}$$

$$= (x-\alpha)^{m-1}\left\{Q'(x)(x-\alpha) + m\,Q(x)\right\}$$

$\therefore$ α is a factor of multiplicity $(m-1)$ 2

(ii) $P(x) = ax^3 + bx^2 + cx + d$

$$P'(x) = 3ax^2 + 2bx + c$$

$$P'(\alpha) = 0$$

$$\alpha = 3a\alpha^2 + 2b\alpha + c$$

IF α IS REAL

$$\alpha = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a}$$

IF REAL

$$4b^2 - 12ac > 0$$

$$4b^2 > 12ac$$

$$b^2 > 3ac \quad 2$$

$$P(\alpha) = 0$$

① $a\alpha^3 + b\alpha^2 + c\alpha + d = 0$

② $3a\alpha^2 + 2b\alpha + c = 0$

α② $3a\alpha^3 + 2b\alpha^2 + c\alpha = 0$

3① $3a\alpha^3 + 3b\alpha^2 + 3c\alpha + 3d = 0$

3① $-\alpha$② $b\alpha^2 + 2c\alpha + 3d = 0$

Hence $\alpha = \dfrac{-2c \pm \sqrt{4c^2 - 12bd}}{2b}$

$$\alpha = \frac{-2c \pm 2\sqrt{c^2 - 3bd}}{2b}$$

$$\alpha = -c \pm \sqrt{c^2 - 3bd} \quad 2$$

(c) $9y^2 + x^2 - 3xy = 6$ (A)

$$\frac{d}{dx}9y^2 + \frac{d}{dx}x^2 - 3\frac{d}{dx}xy = \frac{d}{dx}6$$

$$18y\frac{dy}{dx} + 2x - 3\left(x\frac{dy}{dx} + y\right) = 0$$

$$18y\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 2x$$

$$\frac{dy}{dx} = \frac{3y-2x}{18y-3x}$$

FOR STAT PTS

$$\frac{dy}{dx} = 0 \quad 0 = 3y - 2x$$

$$2x = 3y$$

$$y = \tfrac{2}{3}x$$

SUB INTO (A)

$$9\left(\tfrac{2}{3}x\right)^2 + x^2 - 3x\,\tfrac{2}{3}x = 6$$

$$4x^2 + x^2 - 2x^2 = 6$$

$$3x^2 = 6$$

$$x^2 = 2 \quad (2)$$

$$x = \pm\sqrt{2}$$

(d) S.1. Prove true for $n=1$ $n=2$

$t_1 = 1$ $t_2 = 1$

$$t_1 = \frac{\left(\frac{1+\sqrt5}{2}\right)^1 - \left(\frac{1-\sqrt5}{2}\right)^1}{\sqrt5}$$

$$= \frac{\sqrt5}{\sqrt5} = 1 \quad \text{TRUE FOR } n=1$$

$$t_2 = \frac{\left(\frac{1+\sqrt5}{2}\right)^2 - \left(\frac{1-\sqrt5}{2}\right)^2}{\sqrt5}$$

TWO SQUARES

$$t_2 = \frac{\left(\frac{1+\sqrt5}{2} + \frac{1-\sqrt5}{2}\right)\left(\frac{1+\sqrt5}{2} - \frac{1-\sqrt5}{2}\right)}{\sqrt5}$$

$$t_2 = \frac{(1)\left(2\sqrt5/2\right)}{\sqrt5} = 1$$

TRUE FOR $n=2$

S.2. Assume true for $n=k$ and $n=k+1$ prove true for $n=k+2$

$$T_n = \frac{\alpha^n - \beta^n}{\sqrt5} \quad T_{n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt5}$$

R.T.P.

$$T_{n+2} = \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt5}$$

$$\text{also} = \frac{\alpha^n - \beta^n}{\sqrt5} + \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt5}$$

$$= \frac{\alpha^n(\alpha+1) - \beta^n(\beta+1)}{\sqrt5}$$

$$= \alpha^n\left(\frac{1+\sqrt5}{2} + 1\right) - \beta^n\left(\frac{1-\sqrt5}{2} + 1\right)$$

$$= \frac{\alpha^n\left(\frac{3+\sqrt5}{2}\right) - \beta^n\left(\frac{3-\sqrt5}{2}\right)}{\sqrt5}$$

$$= \frac{\alpha^n\alpha^2 - \beta^n\beta^2}{\sqrt5} \quad (4)$$

$$= \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt5} = T_{n+2}$$

~~Statement~~ holds for T_1 and T_2

S2 implies if true for 1, 2 true for 3, 4, 5 . . .

by the principle of Mathematical Induction true for all n.

Question 7

IN TRIANGLE SXR

$\angle SXR = 90°$

$\therefore$ SR FORMS THE DIAGONAL OF A CIRCLE CENTRE M RADIUS MX

$\therefore$ MX = MS = MR

$\therefore$ ΔMXS, ΔMXR ARE ISOSCELES

Let $\angle XSM = \theta$

$\therefore$ $\angle MXS = \theta$ (ISOS Δ)

AS $\angle SXR = 90°$

$\angle XSM + \angle XRS = 90°$

$\angle XRS = 90 - \theta$

$\therefore$ $\angle RXM = 90 - \theta$ (ISOS Δ)

NOW $\angle SXM = \angle BXQ$ (VERT OPP)

$\angle PRS = \angle PQS$ (SAME ARC SP)

$\therefore$ IN ΔXBQ

$\angle BXQ = \theta$

$\angle PQS = 90 - \theta$

$\therefore$ $\angle XBQ = 90$ (3RD $\angle$ OF Δ)

$\therefore$ MB $\perp$ PQ

4

(b) (i) $I_n = \int_0 x^n \sqrt{1-x}\, dx$

$= \left[x^n \cdot -\frac{2}{3}(1-x)^{3/2} \right]_0^1$

$+ \frac{2}{3} \int_0^1 n x^{n-1} (1-x) \sqrt{1-x}\, dx$

$= 0 + \frac{2n}{3} \int_0^1 x^{n-1}\sqrt{1-x} - x^n \sqrt{1-x}\, dx$

$3I_n = 2n(I_{n-1} - I_n)$

Hence $I_n = \frac{2n}{2n+3} I_{n-1}$.

2

(ii) $I_3 = \frac{6}{9} I_2 = \frac{6}{9} \cdot \frac{4}{7} I_1$

$= \frac{6}{9} \cdot \frac{4}{7} \cdot \frac{2}{5} I_0$

$I_0 = \int_0^1 \sqrt{1-x}\, dx = -\frac{2}{3}\left[(1-x)^{3/2}\right]_0^1$

$= \frac{2}{3}$

$I_3 = \frac{32}{315}$

2

(iii) $I_n = \frac{2n}{2n+3} \cdot \frac{2n-2}{2n+1} \cdot \frac{2n-4}{2n-1} \cdot$

$\times \frac{6}{9} \times \frac{4}{7} \times \frac{2}{5} \cdot \frac{2}{3}$

$= \frac{2^n (n)(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1}{(2n+3)(2n+1) \ldots 7 \cdot 5} \frac{2}{3}$

$= \frac{2^{n+1} n! (2n+2)(2n)(2n-2 \ldots 4 \times 2}{(2n+3)!}$

$= \frac{2^{n+1} n!\, 2^{n+1}\, n+1!}{(2n+3)!}$

$= n!(n+1)! \ldots^{+1}$

3

(c) (i) P(Repeat) $= \frac{24}{100}$

1

(ii) $P(7) = \frac{19}{100}$

1

$P(\bar{7}) = \frac{81}{100}$

(iii) ANY FOUR DIGITS IN ORDER 1 way

But ${}^4P_1 = 24$ way

P(ascending) $= \frac{1}{24}$

2

Question 8.

(a) (i) $x^4+x^3+x^2+x+1 = \frac{x^5-1}{x-1}$

$\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^5$

$= \cos 2\pi + i\sin 2\pi$

$= 1$

$x^5-1 = 0$ (1)

(ii) $b^2+b-1 = \left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-1$

$= x^2+2+\frac{1}{x^2}+x+\frac{1}{x}-1$

$= x^2+x+1+\frac{1}{x}+\frac{1}{x^2}$

$= \frac{1}{x^2}\left(x^4+x^3+x^2+x+1\right)$

$= 0$ (1)

(iii) $x^2-\left(x+\frac{1}{x}\right)x+1$

$= x^2-x^2-1+1$

$= 0$ (1)

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{b \pm \sqrt{b^2-4}}{2}$

$x = \frac{b \pm \sqrt{(4-b^2)(-1)}}{2}$

$x = \frac{b \pm i\sqrt{4-b^2}}{2}$ (1)

$x = \frac{b}{2} + i\frac{\sqrt{4-b^2}}{2}$

$\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5} = \frac{b}{2}+i\frac{\sqrt{4-b^2}}{2}$

equating real parts.

$\cos 2\pi/5 = b/2$

as $b^2+b-1=0$

$b = \frac{-1\pm\sqrt{1+4}}{2}$

$b = \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$

by $b \geq 0$

$\therefore b = \frac{-1+\sqrt{5}}{2}$

$\cos 2\pi/5 = b/2 = \frac{-1+\sqrt{5}}{4}$. (2)

(b) (i) $\int_1^2 \frac{1}{x}dx = \ln x\Big]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2.$ (1)

(ii) Lower rectangles

$A_L = \frac{1}{2}f(1.5)+\frac{1}{2}(f(2))$

$= \frac{1}{2}\frac{2}{3}+\frac{1}{2}\frac{1}{2}$

$= 1/3 + 1/4$

$= \frac{7}{12}$

$A_u = \frac{1}{2}f(1)+\frac{1}{2}f(1.5)$

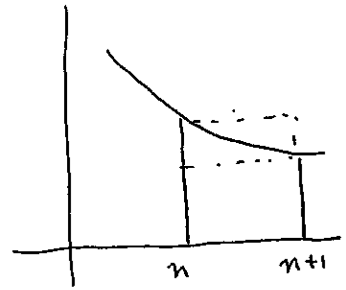
$= \frac{1}{2}+\frac{1}{2}\cdot\frac{2}{3}$

$= \frac{1}{2}+\frac{1}{3}$

$= \frac{5}{6}$

$\therefore \frac{7}{12} \leq \int_1^2 \frac{1}{x}dx \leq \frac{5}{6}$ (1)

(iii)



$A_u = 1\times f(n)$

$= 1\times\frac{1}{n}$

$= \frac{1}{n}$

$A_L = 1\times f(n+1)$

$= 1\times\frac{1}{n+1}$

$= \frac{1}{n+1}$

$\therefore \frac{1}{n+1} \leq \int_n^{n+1}\frac{1}{x}dx \leq \frac{1}{n}$ (1)

Now

$\frac{1}{n+1} \leq \ln\frac{n+1}{n}$

$e^{\frac{1}{n+1}} \leq \frac{n+1}{n}$

$e \leq \left(1+\frac{1}{n}\right)^{n+1}$

also $\ln\frac{n+1}{n} \leq \frac{1}{n}$

$\frac{n+1}{n} \leq e^{\frac{1}{n}}$

$\left(1+\frac{1}{n}\right) \leq e$

$\therefore \left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)$

(c) (i) $\frac{8!}{2!2!2!2!}$ ← 8 letters, 4 repeats

$= 2520$ ways. (1)

(ii) CONSIDER MM as 1 group now 7 groups

$= \frac{7!}{2!2!2!}\times 4$

$= 630\times 4$ ways (1)

$= 2520$ ways

(iii) CONSIDER

TWO PAIRS REPEATED

${}^4C_2 \times \frac{6!}{2!2!} = 1080$

THREE PAIRS REPEATED

${}^4C_3 \times \frac{5!}{2!} = 240$

FOUR PAIRS REPEATED

$4! = 24$

$\therefore$ ARRANGEMENTS WITH 1 OR MORE REPEATS

$2520 - (1080-(240-24))$

$= 1656$

TOTAL # WITHOUT REPETITIO

$2520 - 1656 = 864.$